Topic 2 -

Matrices

HW 2 MATRICES

Def: A <u>matrix</u> is a rectangular array of numbers. If M is a matrix and it has m rows and n columns then we say that M is an $\underbrace{m \times n}_{\text{read: "m by n"}}$ matrix.

Abstractly we can write an $m \times n$ matrix like this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is the entry in the i-th row and j-th column.

Ex:

$$M = \begin{pmatrix} 1 & 5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$a_{11} = 1$

$a_{12} = 5$

$a_{21} = 3$

$a_{22} = -2$

M is a 2×2 matrix.

Ex:

$$M = \begin{pmatrix} 5 & 0 & 10 & 7 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

M is a 1×4 matrix.

$a_{11} = 5$

$a_{12} = 0$

$a_{13} = 10$

$a_{14} = 7$

Note: You can use commas if you want to make it clearer. Like this:

$$M = (5, 0, 10, 7)$$

Note: Sometimes we want to think of a vector as a matrix.

Suppose we have $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ in $\mathbb{R}^n$.

We can think of $\vec{v}$ as an $n \times 1$ matrix $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

or we can think of $\vec{v}$ as a $1 \times n$ matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$

Ex: $\vec{v} = \langle 1, 5, \frac{1}{2} \rangle$

Can think of $\vec{v}$ as $\begin{pmatrix} 1 \\ 5 \\ 1/2 \end{pmatrix}$ 3x1 matrix

or $\begin{pmatrix} 1 & 5 & \frac{1}{2} \end{pmatrix}$. 1x3 matrix

Def: Let A and B be $m \times n$ matrices.
[They have the same size.]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

1. We define $A+B$ to be the following $m \times n$ matrix:

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

2 We define $A-B$ to be the following $m \times n$ matrix:

$$A+B = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

3 If α is in $\mathbb{R}$, the scalar product αA is defined to be the $m \times n$ matrix:

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Ex:

$$\underbrace{\begin{pmatrix} 0 & 5 \\ 3 & 1 \end{pmatrix}}_{2\times 2} + \underbrace{\begin{pmatrix} 2 & -1 \\ 6 & 7 \end{pmatrix}}_{2\times 2} = \begin{pmatrix} 0+2 & 5-1 \\ 3+6 & 1+7 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 9 & 8 \end{pmatrix}$$

same size ✓

Ex:

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{4\times 1} - \underbrace{\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}}_{4\times 1} = \begin{pmatrix} 1-2 \\ 1-3 \\ 1-4 \\ 1-5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix}$$

same size ✓

Ex:

$$\underbrace{\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}}_{2\times 2} + \underbrace{\begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 1 & 8 \end{pmatrix}}_{3\times 2}$$

this sum is undefined since the matrices don't have the same size.

Ex:

$$10\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 30 \\ -10 & 50 \end{pmatrix}$$

Def: Let A be an $m \times r$ matrix and B be an $r \times n$ matrix. We define the **product** of A and B, denoted by AB, as the $m \times n$ matrix C whose entry at row i and column j is defined to be the dot product of the i-th row of A and the j-th column of B.

$$\underbrace{C}_{m \times n} = \underbrace{A}_{m \times r} \; \underbrace{B}_{r \times n}$$

must equal

Ex: Calculate AB, if possible, where

$$A = \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}}_{2\times 2} \quad \text{and} \quad B = \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{2\times 3}$$

equal, so AB is defined

AB will be 2×3

(row 1 of A)$\cdot$(column 1 of B)   (row 1 of A)$\cdot$(column 2 of B)   (row 1 of A)$\cdot$(column 3 of B)

$$AB = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix}$$

(row 2 of A)$\cdot$(column 1 of B)   (row 2 of A)$\cdot$(column 2 of B)   (row 2 of A)$\cdot$(column 3 of B)

$$= \begin{pmatrix} (1)(1)+(2)(0) & (1)(2)+(2)(1) & (1)(-1)+(2)(0) \\ (-1)(1)+(0)(0) & (-1)(2)+(0)(1) & (-1)(-1)+(0)(0) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 4 & -1 \\ -1 & -2 & 1 \end{pmatrix}}_{AB}.$$

Ex: Using the same matrices can we calculate BA?

$$\underbrace{B}_{2\times 3}\ \underbrace{A}_{2\times 2}$$

$3 \neq 2$

Since $3 \neq 2$, BA is not defined.

You can also see this if you tried to multiply them.

$$BA = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

(row 1 of B)·(column 1 of A)

$$= \begin{pmatrix} \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \cdots \\ \cdots & \end{pmatrix}$$

You can't do this dot product since the sizes aren't the same.

<u>Ex:</u> Let

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}.$$

Calculate $\underbrace{A}_{3\times 1}\underbrace{B}_{1\times 3}$ if possible.

✓

answer is 3×3

$$AB = \begin{pmatrix}
\begin{matrix}\text{(row 1 of A)}\cdot \\ \text{(column 1 of B)} \\ (1)(0)\end{matrix} &
\begin{matrix}\text{(row 1 of A)}\cdot \\ \text{(column 2 of B)} \\ (1)(1)\end{matrix} &
\begin{matrix}\text{(row 1 of A)}\cdot \\ \text{(column 3 of B)} \\ (1)(-3)\end{matrix} \\
\begin{matrix}\text{(row 2 of A)}\cdot \\ \text{(column 1 of B)} \\ (2)(0)\end{matrix} &
\begin{matrix}\text{(row 2 of A)}\cdot \\ \text{(column 2 of B)} \\ (2)(1)\end{matrix} &
\begin{matrix}\text{(row 2 of A)}\cdot \\ \text{(column 3 of B)} \\ (2)(-3)\end{matrix} \\
\begin{matrix}\text{(row 3 of A)}\cdot \\ \text{(column 1 of B)} \\ (-1)(0)\end{matrix} &
\begin{matrix}\text{(row 3 of A)}\cdot \\ \text{(column 2 of B)} \\ (-1)(1)\end{matrix} &
\begin{matrix}\text{(row 3 of A)}\cdot \\ \text{(column 3 of B)} \\ (-1)(-3)\end{matrix}
\end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & -6 \\ 0 & -1 & 3 \end{pmatrix}$$

Ex: Let

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}$$

as before. Can we calculate BA?

$$BA = \underbrace{\begin{pmatrix} 0 & 1 & -3 \end{pmatrix}}_{1 \times 3} \cdot \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{3 \times 1}$$

✓

BA is 1x1

$$\begin{pmatrix} \text{row } 1 \\ \text{of } B \end{pmatrix} \cdot \begin{pmatrix} \text{column } 1 \\ \text{of } A \end{pmatrix}$$

$$= \left(\begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right)$$

$$= \left((0)(1) + (1)(2) + (-3)(-1) \right)$$

$$= \left(5 \right)$$

$\boxed{BA \text{ is } 1 \times 1}$

__Note:__

In the previous examples when $A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix}$

we saw that $AB \neq BA$

* *

$$\boxed{\text{In general, } AB = BA \text{ is } \underline{\underline{\text{NOT}}} \text{ always true for matrices}}$$

Def: Let A be an $m \times n$ matrix. The **transpose** of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows of columns of A.

That is, the i-th column of A^T is the i-th row of A.

[Similarly, the j-th row of A^T is the j-th column of A.]

Some people write A^t instead of A^T

Ex: Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \quad 3 \times 4$$

$$A^T = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix} \quad 4 \times 3$$

You could also have turned the columns of A into the rows of A^T and you'd get the same answer

Ex:

$$A = \begin{pmatrix} 1 & -1 & \pi \\ \sqrt{2} & 1/2 & 5 \\ 10 & 7 & -2 \end{pmatrix} \qquad 3 \times 3$$

$$A^T = \begin{pmatrix} 1 & \sqrt{2} & 10 \\ -1 & 1/2 & 7 \\ \pi & 5 & -2 \end{pmatrix} \qquad 3 \times 3$$

Def: The $m \times n$ zero matrix is the $m \times n$ matrix where every entry is zero. We denote it by $O_{m\times n}$ or just by O if we don't want to mention the size.

Ex:

$$O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad O_{4\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$O_{5\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex:

Let $A = \begin{pmatrix} 1 & 5 \\ 7 & 2 \\ 3 & -1 \end{pmatrix}$

Then,

$$A + O_{3\times 2} = \begin{pmatrix} 1 & 5 \\ 7 & 2 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 \\ 7 & 2 \\ 3 & -1 \end{pmatrix} = A$$

Similarly,

$$O_{3\times 2} + A = A$$

Def: The $n \times n$ identity matrix, denoted by I_n [or just I when we don't want to or need to say the size], is the $n \times n$ matrix with 1's along the main diagonal and 0's everywhere else.

<u>Ex:</u>

$$I_1 = \begin{pmatrix} 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so on.

<u>Ex:</u> Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \leftarrow \boxed{2\times 2}$

Consider

$$I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have that

$$AI = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2\times 2}$$

equal ✓

answer is 2×2

$$= \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1)+(2)(0) & (1)(0)+(2)(1) \\ (3)(1)+(4)(0) & (3)(0)+(4)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

So, $A I_2 = A$.

You can also calculate

$\underbrace{I_2}_{2\times 2} \underbrace{A}_{2\times 2}$ which is defined

and you will get $I_2 A = A$.

Ex:

$$\text{Let } A = \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} \leftarrow \boxed{2 \times 3}$$

Note that

$$\underbrace{I_2}_{2\times 2} \underbrace{A}_{2 \times 3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix}$$

✓ answer = 2x3

$$= \begin{pmatrix} (1\ 0) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & (1\ 0) \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} & (1\ 0) \cdot \begin{pmatrix} \pi \\ -2 \end{pmatrix} \\ (0\ 1) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & (0\ 1) \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} & (0\ 1) \cdot \begin{pmatrix} \pi \\ -2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} = A$$

Note that $\underbrace{A}_{2\times 3}\ \underbrace{I_2}_{2\times 2}$ is not defined.

$3 \neq 2$

But if you calculate

$$\underbrace{A}_{2\times 3}\ \underbrace{I_3}_{3\times 3} = \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} = A$$

you fill in this part

So, $AI_3 = A$.

Theorem:

Let A, B, C be matrices and let α, β be real numbers.

α = alpha
β = beta

Then the following are true where we will assume that the sizes of the matrices are such that the operations are defined:

1. $A+B = B+A$
2. $A+(B+C) = (A+B)+C$
3. $A(BC) = (AB)C$
4. $A(B+C) = AB+AC$
5. $(B+C)A = BA+CA$
6. $A(B-C) = AB-AC$

7. $(B-C)A = BA - CA$
8. $\alpha(B+C) = \alpha B + \alpha C$
9. $\alpha(B-C) = \alpha B - \alpha C$
10. $(\alpha+\beta)A = \alpha A + \beta A$
11. $(\alpha-\beta)A = \alpha A - \beta A$
12. $\alpha(\beta A) = (\alpha\beta)A$
13. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
14. $(A^T)^T = A$
15. $(A+B)^T = A^T + B^T$
16. $(A-B)^T = A^T - B^T$
17. $(\alpha A)^T = \alpha A^T$
18. $(AB)^T = B^T A^T$

note the reversal of the order

19. If A is $m \times n$, then

$$A I_n = A.$$

20. If A is $m \times n$, then

$$I_m A = A$$

21. If A is $m \times n$, then

$$A - A = O_{m \times n}$$

22. If A is $m \times n$, then

$$A + O_{m \times n} = O_{m \times n} + A = A$$

Lets prove part 5 of the previous theorem for 2x2 matrices
[This is HW 2-Part 2 #1(a)]

Suppose that A, B, C are 2×2 matrices.

Given

Prove that
$$(B+C)A = BA + CA$$

what you need to show

<u>proof:</u>
Let A, B, C be 2×2 matrices.

declare what the objects are

Then,
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},\ C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$
where a,b,c,d,e,f,g,h,i,j,k,l are real numbers.

Then,

$$(B+C)A = \underbrace{\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right]}_{\text{do first because of parentheses}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}$$

$2 \times 2 \quad \checkmark \quad 2 \times 2$

answer is 2×2

$$= \begin{pmatrix} (e+i \quad f+j)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (e+i \quad f+j)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (g+k \quad h+l)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (g+k \quad h+l)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (e+i)a+(f+j)c & (e+i)b+(f+j)d \\ (g+k)a+(h+l)c & (g+k)b+(h+l)d \end{pmatrix}$$

$$= \begin{pmatrix} ea+ia+fc+jc & eb+ib+fd+jd \\ ga+ka+hc+lc & gb+kb+hd+ld \end{pmatrix} \quad (*)$$

We also have that

$$BA+CA = \underbrace{\begin{pmatrix} e & f \\ g & h \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2} + \underbrace{\begin{pmatrix} i & j \\ k & l \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}$$

✓ answer is 2x2 ✓ answer is 2x2

$$= \begin{pmatrix} (e\ f)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (e\ f)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (g\ h)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (g\ h)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} (i\ j)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (i\ j)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (k\ l)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (k\ l)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix} + \begin{pmatrix} ia+jc & ib+jd \\ ka+lc & kb+ld \end{pmatrix}$$

$$= \begin{pmatrix} ea+fc+ia+jc & eb+fd+ib+jd \\ ga+hc+ka+lc & gb+hd+kb+ld \end{pmatrix} \quad (**)$$

We can see that $(*)$ equals $(**)$.

Thus,

$$(B+C)A = BA + CA. \quad \blacksquare$$

other ways:
QED, $\square$

(end of proof symbol)

Let's prove part 15 for 3×2 matrices.
[This is HW 2 - Part 2 #1(f)]

Let A and B be 3×2 matrices. — Given

Prove that $(A+B)^T = A^T + B^T$ — Need to show

<u>Proof:</u>

Let A and B be 3×2 matrices. — declare the objects

Then,

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \text{ and } B = \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}$$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are real numbers.

We have that

$$(A+B)^T = \Bigg(\underbrace{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} + \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}}_{\text{add first because of parentheses}} \Bigg)^T$$

$$= \begin{pmatrix} a+g & b+h \\ c+i & d+j \\ e+k & f+l \end{pmatrix}^T = \begin{pmatrix} a+g & c+i & e+k \\ b+h & d+j & f+l \end{pmatrix}$$

$$= \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} + \begin{pmatrix} g & i & k \\ h & j & l \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}^T + \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}^T$$

$$= A^T + B^T \qquad \blacksquare$$

HW 2 - Part 2

② (a) Suppose that A, B, C, D are $n \times n$ matrices. Use the properties from class to show that

$$(A+B)(C+D) = AC + AD + BC + BD$$

<u>proof:</u> Let A, B, C, D be $n \times n$ matrices. Then,

$$\underbrace{\underbrace{(A+B)}_{n\times n}\underbrace{(C+D)}_{n\times n}}_{\text{ok}} = \overbrace{\overbrace{(A+B)}^{n\times n}\overbrace{C}^{n}}^{\text{ok}} + \overbrace{\overbrace{(A+B)}^{n\times n}\overbrace{D}^{n\times n}}^{\text{ok}}$$

$X(C+D) = XC + XD$
property ④ from class
use $X = A+B$

$$= AC + \underbrace{BC + AD}_{} + BD$$

$(M+N)X = MX + NX$
property ⑤ from class

prop. ①
$X + Y = Y + X$

$$= AC + \underbrace{AD + BC}_{} + BD$$

$\boxed{\diagup\diagup}$